



A Short Note on Hyper-geometric Expression

J. López-Bonilla, V. M. Salazar del Moral

ESIME-Zacatenco, National Polytechnic Institute, Col. Lindavista CP 07738, CDMX, México

Corresponding author: jlopezb@ipn.mx

Received: May 20, 2016

Revised: July 25, 2016

Accepted: July 28, 2016

Abstract: In this short note, we exhibit an elementary deduction of the Borwein-Choi-Pigulla relation for $\log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k}$ in terms of the Gauss hyper-geometric function.

Keywords: Hyper-geometric function, Borwein-Choi-Pigulla's formula.

1. Introduction

In this paper, we briefly discuss on the hyper-geometric expression of $\log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k}$. The researchers Borwein-Choi-Pigulla [3] employed continued fractions to obtain the following identity [2]:

$$\log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} = -\frac{(-z)^n}{n} {}_2F_1(n, 1; n+1; -z), \quad n = 2, 3, \dots \quad (1)$$

Here we show a simple deduction of (1) via the techniques of [4, 5] to convert a summation into a hyper-geometric function.

2. Borwein-Choi-Pigulla Formula

We have the known Taylor series for the logarithm function:

$$\begin{aligned} \log(1+z) - z &= -\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots = -\frac{z^2}{2} \left(1 - \frac{2z}{3} + \frac{2z^2}{4} - \frac{2z^3}{5} + \dots \right), \\ &= -\frac{z^2}{2} \sum_{k=0}^{\infty} u_k, \quad u_k = \frac{2(-1)^k z^k}{k+2}, \end{aligned} \quad (2)$$

and it is simple to apply the techniques of [4, 5] to convert a summation into a Gauss hyper-geometric function, in fact,

$$u_0 = 1 \text{ \& } \frac{u_{k+1}}{u_k} = \frac{(k+2)(k+1)(-z)}{(k+3)(k+1)}, \text{ therefore, we have}$$

$$\log(1+z) - z = -\frac{z^2}{2} {}_2F_1(2, 1; 3; -z).$$

Similarly, from (2):

$$\begin{aligned}\log(1+z) - z + \frac{z^2}{2} &= \frac{z^3}{3} \sum_{k=0}^{\infty} \frac{3(-1)^k z^k}{k+3} \\ &= \frac{z^3}{3} {}_2F_1(3, \quad 1; 4; -z)\end{aligned}$$

and in general:

$$\begin{aligned}\log(1+z) + \sum_{k=1}^{n-1} \frac{(-z)^k}{k} &= -\frac{(-z)^n}{n} \sum_{k=0}^{\infty} \frac{n(-z)^k}{k+n} \\ &= -\frac{(-z)^n}{n} {}_2F_1(n, 1; n+1; -z)\end{aligned}$$

in harmony with the result (1) due to Borwein-Choi-Pigulla.

As always, a formula for *log* leads correspondingly to one for *arc tan* [2]:

$$\arctan z - \sum_{k=0}^{n-1} \frac{(-1)^k z^{2k+1}}{2k+1} = \frac{(-1)^n z^{2n+1}}{2n+1} {}_2F_1\left(n + \frac{1}{2}, 1; n + \frac{3}{2}; -z^2\right). \quad (3)$$

3. Conclusion

Our procedure shows that the techniques of [4, 5] are very useful to translate a summation in terms of hyper-geometric functions, and thus to give simple proofs for several important formulae in the literature [1].

References

- [1] Barrera FV, López-Bonilla J and López-Vázquez R (2016), On the Bailey-Borwein-Plouffe formula for π . *Prespacetime Journal* **7**(10) : 1408-1410.
- [2] Borwein J, Bailey D and Girgensohn R (2004), *Experimentation in Mathematics. Computational paths to discovery*. A. K. Peters, Natick, Mass. USA.
- [3] Borwein JM, Choi S and Pigulla W (2005), Continued fractions of tails of hypergeometric series. *Am. Math. Monthly*, Dec.
- [4] Koepf W (1998), *Hypergeometric summation. An algorithmic approach to summation and special function identities*. Vieweg, Braunschweig/Wiesbaden.
- [5] Petkovsek M, Wilf HS and Zeilberger D (1996), *A = B, symbolic summation algorithms*. A. K. Peters, Wellesley, Mass.